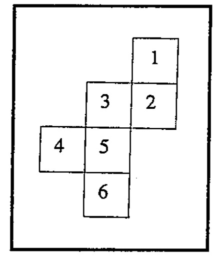
<u>*MCWP Senior High Mathematics League*</u>

Contest 1 **October 19, 2009**

Individual Questions

-- CUT HERE --

INDIVIDUAL ROUND 1 — 15 MINUTES



I–1. When this net of six squares is cut out and folded to form a cube, what is the product of the numbers on the four faces adjacent to the face labeled with a 1?

I–2. If the numbers in the list, $2^1,\ 2^2,\ 2^3,\ 2^4, \cdots,\ 2^{100}$ are written in ordinary base-ten notation (for example: $2,\ 4,\ 8,\ 16, \cdots$) how many of the numbers will have a units' digit of 6?

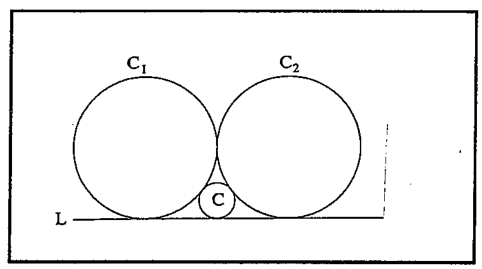
I–3. The 5 letters in the word, PINTO, can be arranged 120 ways. If these 120 permutations are arranged in dictionary order, as if each were an ordinary word, find the 85-th "word" in the list. (For example, the first "word" in the list would be INOPT, the second would be INOTP, the third would be INPOT, etc.)

-- CUT HERE --

INDIVIDUAL ROUND 2 — 15 MINUTES

I–4. The coordinates of the vertices of a parallelogram are $(10,1),\ (7,-2),\ (4,1),\ and\ (x,y)$. What is the <u>**sum**</u> of all the distinct possible values for x?

I–5. Circle C is tangent to line L. Two circles C_1 *and* C_2 of equal radii are each tangent to one another, to C and to L. If the radius of C is 3, then what is the radius of C_1 ?



I–6. Sophie has only quarters, dimes and nickels in her piggy bank. There are 60 coins altogether in the bank. If the quarters were dimes and the dimes were quarters, the total value of the coins would be increased by 90 cents. Also, if the nickels were dimes and the dimes were nickels, the total value of the coins would be increased by 15 cents. What is the total value, in cents, of the coins in Sophie's piggy bank?

MCWP Senior High Mathematics League

Contest #1 **October 19, 2009**

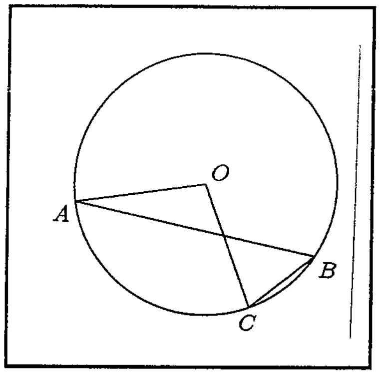
TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. A region is bounded by the y-axis and the line $x = d - 9|y|$ for $d > 0$. For what value of d is the area of the region equal to 36?

T-2. Four whole numbers, when added three at a time, give the sums 180, 197, 208, and 222. What is the largest of the four numbers?

T-3. In the diagram, O is the center of the circle, $\angle OAB = 20°$ and $\angle OCB = 70°$. Find the degree measure of $\angle ABC$.



T-4. Mr. Earl E. Byrd leaves his house for work every morning at exactly 8:00 AM. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Byrd drive to arrive at his workplace precisely on time?

T-5. Let a and b be distinct real numbers for which $\dfrac{a}{b} + \dfrac{a+10b}{b+10a} = 2$. Find the value of $\dfrac{a}{b}$.

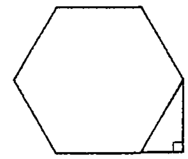
MCWP Senior High Mathematics League

Contest #2 **INDIVIDUAL QUESTIONS** **November 16, 2009**

cut here - cut here

INDIVIDUAL ROUND 1 — 15 MINUTES

1. The hypotenuse of a 30°-60°-90° triangle is one side of a regular hexagon, as shown. If the length of the shorter leg of the right triangle is 1, what is the perimeter of the regular hexagon?



2. What is the smallest positive integer greater than 3 which leaves a remainder of 3 when divided by each of 4, 5, 6, 7, and 8?

3. What value of a satisfies $27x^3 - 16\sqrt{2} = (3x - 2\sqrt{2})(9x^2 + 6x\sqrt{2} + a)$ for all real x?

cut here - cut here

INDIVIDUAL ROUND 2 — 15 MINUTES

4. If the first 25 positive integers are multiplied together, in how many zeroes does the product terminate?

5. What is the smallest positive number x for which $(16^{\sqrt{2}})^x$ represents a positive integer?

6. Of the pairs of positive integers (x,y) that satisfy $3x+7y = 188$, which ordered pair has the least positive difference $y-x$?

cut here - cut here

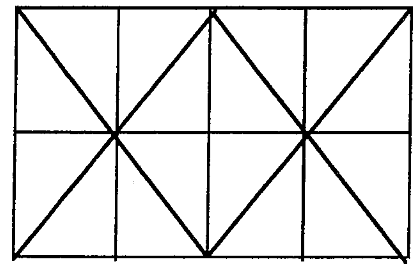
MCWP Senior High Mathematics League

Contest #2 **November 16, 2009**

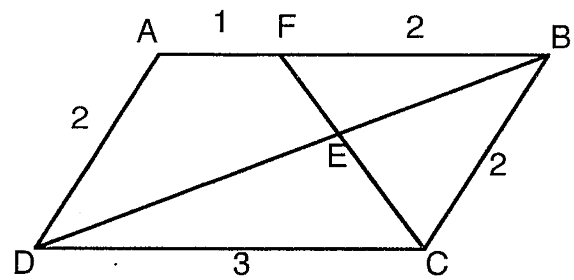
TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. How many triangles are in the figure shown?



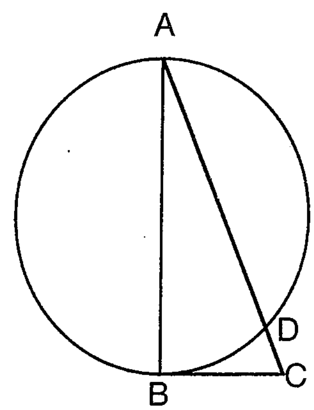
T-2. Find the ratio of the area of the triangle, ΔDEC to the area of the parallelogram, $ABCD$



T-3. Two positive integers are chosen at random. What is the probability that their product is odd? Express your answer as a proper fraction.

T-4. Consider the string of letters, *baacd*. How many other distinct strings are possible using these 5 letters?

T-5. Suppose $\overline{AB}$ is the diameter of the circle shown, $\overline{BC}$ is tangent to the circle, $\angle BAC = 30°$, and $CD = \sqrt{3}$, where D is the point at which $\overline{AC}$ intersects the circle. What is the length of $\overline{AB}$?



MCWP Senior High Mathematics League

Contest #3 **INDIVIDUAL QUESTIONS** **December 14, 2009**

cut here - cut here - cut here

INDIVIDUAL ROUND 1 — 15 MINUTES

1. $\{F_n\}$ is defined by $F_1 = -1, F_2 = 1$, and, for $n > 2, F_n = F_{n-1}+F_{n-2}$. What is the value of F_{10}?

2. The Parallelogram Law says that the sum of the squares of the lengths of all sides of a parallelogram equals the sum of the squares of the lengths of its diagonals. In parallelogram P, both diagonals have integral lengths. If adjacent sides of P have lengths 4 and 7, what is the greatest possible length of one of P's diagonals?

3. What is the product of the greatest common divisor and the least common multiple of two positive integers x and y? (Give your answer in terms of x and y.)

cut here - cut here - cut here

INDIVIDUAL ROUND 2 — 15 MINUTES

4. What is the smallest positive integer greater than 5 which leaves a remainder of 5 when divided by each of 6, 7, 8, and 9?

5. What are all ordered pairs of positive numbers (x,y) for which $x = \sqrt{2y}$ and $y = \sqrt{x}$?

6. How many minutes past 4 o'clock are the hands of a standard 12-hour clock first perpendicular to each other?

cut here - cut here - cut here

MCWP Senior High Mathematics League

Contest #3 **December 14, 2009**

TEAM QUESTIONS

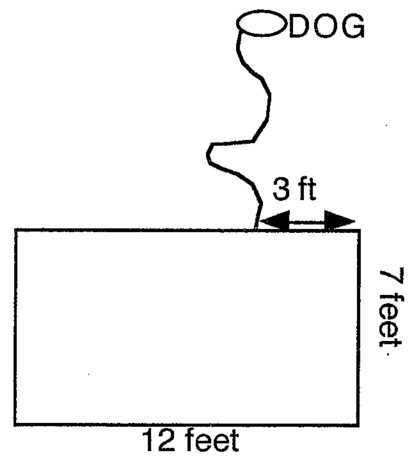
TIME LIMIT = 10 MINUTES

T-1. Find all solutions, x, for which $3^x - 3^{-x} = \frac{80}{9}$

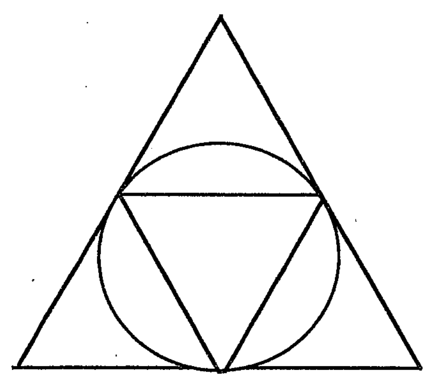
T-2. Find the sum of the first twenty odd numbers that have a remainder of one when divided by three.

T-3. What is the last digit of the number 777^{778} ?

T-4. My dog is staked to a point 3 feet from the corner of my shed, which measures $7ft \times 12ft$. His chain is 13 feet long. Suppose he pulls tightly on the chain, through what area can his leash sweep?



T-5. What is the ratio of the area of an equilateral triangle that circumscribes a circle to the area of an equilateral triangle that is inscribed in the same circle?



MCWP Senior High Mathematics League

Contest #4 INDIVIDUAL QUESTIONS January 11, 2010

cut here - cut here - cut here

INDIVIDUAL ROUND 1 — 15 MINUTES

1. What is the smallest positive prime p for which $p^2 - p - 1$ is a composite number?

2. How many minutes apart are the first two times after 4 o'clock that the hands of a standard 12-hour clock are perpendicular to each other?

3. What are all real values of x which satisfy $|3x| + |x+1| = 2$?

cut here - cut here - cut here

INDIVIDUAL ROUND 2 — 15 MINUTES

4. What is the area of a circle in which a chord of length 6 is twice as far from the center as a chord of length 12?

5. If $x + y = 5$ and $xy = 3$, what is the value of $x^2 + y^2$?

6. What real value of x satisfies $4^{x+2} - 4^x = 30$?

cut here - cut here - cut here

MCWP Senior High Mathematics League

Contest #4 **January 11, 2010**

TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. Let AB be a line segment of length 1. Let C be a point on the line segment such that the ratio of the length of AC to that of CB is the same as the ratio of the length of AB to that of AC. Find this common ratio and write it in its proper form.

T-2. The sum of the third and fourth terms in a sequence of consecutive integers is 47. Find the sum of the first five terms of the sequence.

T-3. Find the area of the triangle whose sides lie on the lines,

$$y=-1,\quad y=-\frac{5}{6}x+\frac{7}{3},\quad \text{and}\quad y=\frac{5}{6}x-1$$

T-4. Let P be the point $(3,\ 2)$. Let Q be the reflection of P across the x-axis, let R be the reflection of Q about the line $y=-x$, and let S be the reflection of R through the origin. What is the area of the quadrilateral $PQRS$?

T-5. A box contains 3 fair coins and 7 biased coins. Whenever a fair coin is flipped, it comes up heads with a probability of 0.5. Whenever a biased coin is flipped, it comes up heads with a probability of 0.6. A coin is randomly chosen from the box and then flipped. What is the probability that it will come up heads?

MCWP Senior High Mathematics League

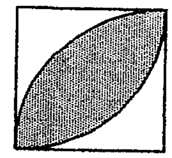
Contest #5 **INDIVIDUAL QUESTIONS** **February 8, 2010**

cut here - cut here - cut here

INDIVIDUAL ROUND 1 — 15 MINUTES

1. What is the numerical value of $\log_{2009}2009$?

2. In the diagram, the arcs of two congruent circles intersect. and the region between them is shaded. The circles are centered at opposite vertices of a square whose side is a radius of each circle. If the area of the square is 4, what is the area of the shaded region?



3. What is the only quadrant that has no points on the graph of $(x-8)^2 + (y+6)^2 = 9^2$?

cut here - cut here - cut here

INDIVIDUAL ROUND 2 — 15 MINUTES

4. If $2009^x + 2009^y = 9$ and $2009^{x+y} = 8$, what is the value of $2009^{2x} + 2009^{2y}$?

5. For what ordered pair of integers (A,B) is x^3+7x^2+Ax+B divisible by x^2+1?

6. Altogether how many different handshakes would occur if each person in a group of 20 people shakes hands once with every other person in the group?

cut here - cut here - cut here

MCWP Senior High Mathematics League

Contest #5 **February 8, 2010**

TEAM QUESTIONS

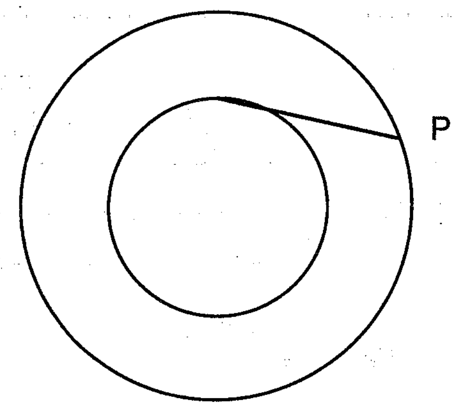
TIME LIMIT = 10 MINUTES

T-1. How many other four-digit integers can be formed by rearranging the digits of 8199?

T-2. Solve the inequality $\dfrac{2x+5}{|x+1|} \geq 1$

T-3. If $x+y+z=0$ and $xyz=2$, then what is the value of $x^3+y^3+z^3$?

T-4. Two concentric circles are such that the small one divides the larger one onto two regions of equal area. If the radius of the smaller circle is 3, compute the length of a tangent from any point P on the larger circle to the smaller circle.



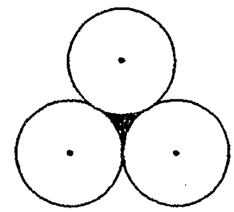
T-5. Albert spent all his money in five stores. In each store, he spent \$1 more than half of what he had when he entered that store. How much money did Albert have when he entered the first store?

MCWP Senior High Mathematics League

Contest #6 **INDIVIDUAL QUESTIONS** **March 8, 2010**

cut here - cut here - cut here

INDIVIDUAL ROUND 1 — 15 MINUTES



1. Three coplanar circles of area π are externally tangent as shown. What is the area of the shaded region?

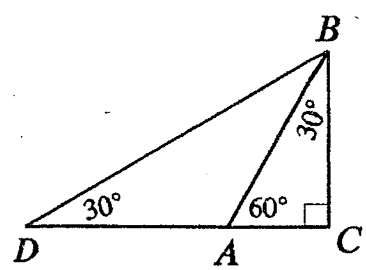
2. What ordered pair of real numbers (x,y) satisfies $269x + 231y = 288$ and $231x + 269y = 212$?

3. I averaged 4 feet per second when I rode my bike from Here to There. Returning over the same route, I averaged k feet per second. For what value of k would my average speed for the round trip have been 6 feet per second?

cut here - cut here - cut here

INDIVIDUAL ROUND 2 — 15 MINUTES

4. A certain escalator (moving stairway) always moves at a constant rate. A man walked up this escalator at a constant rate, his speed adding 3 steps per second to the speed of the escalator. The trip from bottom to top took 10 seconds. He then walked down the same *up-moving* escalator at his constant rate of 3 steps per second. The trip from top to bottom took 50 seconds. How many steps-per-second is each step of the escalator moving?



5. As shown, $\triangle DBC$ and $\triangle ABC$ are 30°-60°-90° right triangles. If $DA = 100$, what is BC?

6. Simplify $(\log_{16} 9)(\log_3 25)(\log_5 4)$ as much as possible.

cut here - cut here - cut here

MCWP Senior High Mathematics League

Contest #6 **March 8, 2010**

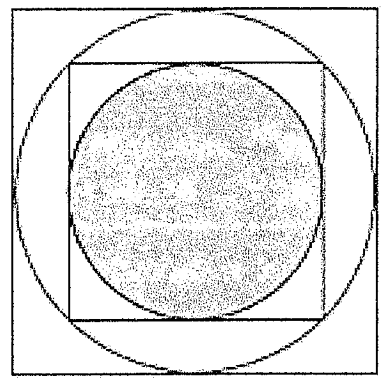
TEAM QUESTIONS

TIME LIMIT = 10 MINUTES

T-1. If the product of three numbers in geometric progression is 216 and their sum is 19, find the largest of the three numbers.

T-2. What is the sum of the digits of $10^{55} - 55$?

T-3. A circle is inscribed in a square, that is inscribed in a circle, that is inscribed in a square. (See the diagram.) The area of the largest square is 108 square inches. Find the area of the shaded circle.



T-4. Find the sum of the first 19 terms of an arithmetic sequence, $a_1, a_2, a_3, \ldots, a_{19}$ if it is known that $a_4 + a_8 + a_{12} + a_{16} = 224$.

T-5. Of all the combinations of positive integers that sum to 10, what is their largest possible product?

SOLUTIONS TO INDIVIDUAL QUESTIONS

Contest #1, 2009-2010 **October 19, 2009**

I–1. Visualize the cube with the top face labeled with a "1". As you imagine folding the squares into a cube, you can see that the face labeled "5" will be on the bottom. Therefore the four sides adjacent to the "1" face will contain the numbers $2, 3, 4$, and 6, and the required product is 144.

I–2. Every power of 2^4 ends in a 6. Since $2^{100} = \left(2^4\right)^{25}$, then there are 25 at least.

Now consider k so that 4 does not divide k, so $k = 4s + t$, where $t = 1,\ 2,\ or\ 3$ and $1 \le k \le 100$.

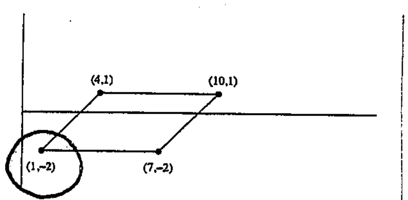
Then $2^k = 2^{4s+t} = \left(2^{4s}\right)\left(2^t\right),\ where\ 2^t = 2,\ 4,\ or\ 8$

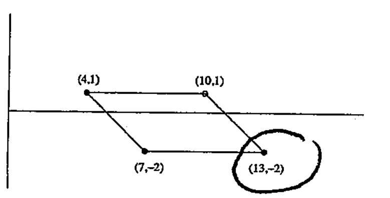
Now, 2^{4s} has a units' digit of 6, but $\left(2^{4s}\right)(2),\ \left(2^{4s}\right)(4),\ and\ \left(2^{4s}\right)(8)$ do not.

So, there are no other powers of 2 that end in 6.

I–3. There are 24 "words beginning with the letter I, then 24 beginning with N, and 24 beginning with O. That accounts for 72 "words". We now have PI _ _ _, 6 of these, and PN _ _ _, 6 of these. This gets us to 84 "words". Then the 85-th "word" is POINT.

I–4. Since three points are given, there are three possibilities for the 4-th point. Two are easily found:



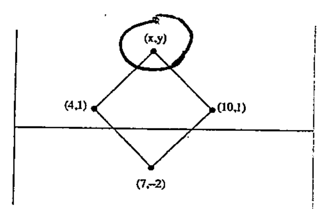


For the 3rd point, we have from equality of slopes:

$$\frac{y-1}{x-10} = \frac{7-4}{-2-1} = -1$$
$$\frac{y-1}{x-4} = \frac{-3}{-3} = 1$$

$$-(x-10) = x-4$$
$$x = 7$$

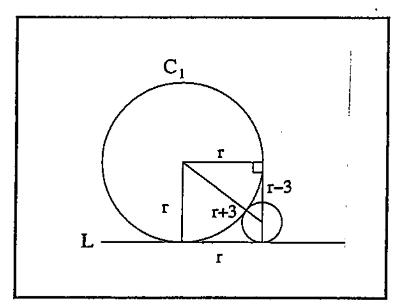
Therefore, the sum is $1 + 13 + 7 = 21$.



I–5. In the diagram, $r^2 + (r-3)^2 = (r+3)^2$.

Then $2r^2 - 6r + 9^2 = r^2 + 6r + 9$

And $r^2 - 12r = 0 \ \Rightarrow\ r(r-12) = 0 \ \Rightarrow\ r = 12$



I–6. Let $q, d,$ and n represent, respectively, the number of quarters, dimes and nickels in her piggy bank.

Then $\begin{cases} q+d+n = 60 \\ 10q + 25d + 5n = 25q + 10d + 5n + 90 \\ 25q + 5d + 10n = 25q + 10d + 5n + 15 \end{cases} \Rightarrow \begin{cases} q+d+n = 60 \\ q - d = -6 \\ d - n = -3 \end{cases}$

The solution of the above is $q = 15,\ d = 21,\ and\ n = 24$. So the total value of the coins, in cents, is $25q + 10d + 5n = 25(15) + 10(21) + 5(24) = 705$ cents.

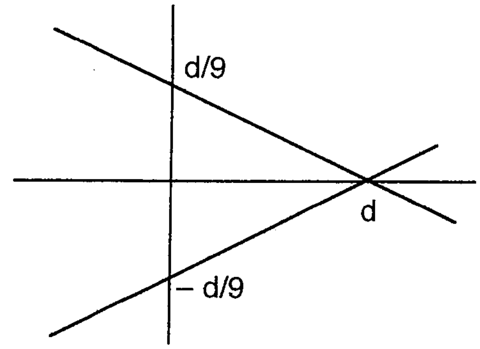
SOLUTIONS TO TEAM QUESTIONS

Contest #1, 2009 - 2010 **October 19, 2009**

T-1. If $y \ge 0$, then $x = d - 9|y| = d - 9y \Rightarrow y = -\frac{1}{9}x + \frac{d}{9}$

And, if $y < 0$, then $x = d - 9|y| = d + 9y \Rightarrow y = \frac{1}{9}x - \frac{d}{9}$

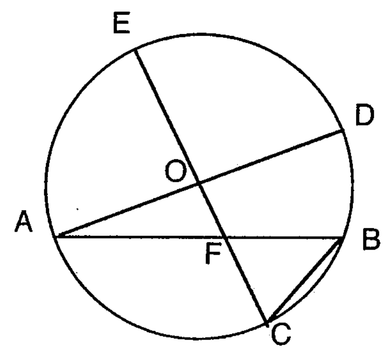
These two lines intersect at $(d,0)$. Therefore, the triangle has area $\left(\frac{d}{9}\right)(d) = 36 \Rightarrow d^2 = 9 \cdot 36 \Rightarrow d = 3 \cdot 6 = 18$



T-2. Let $\begin{cases} a+b+c=180 \\ a+b+d=197 \\ a+c+d=208 \\ b+c+d=222 \end{cases}$ Then, adding the 4 equations we get: $3(a+b+c+d)=807$ *so that* $a+b+c+d=269$

Subtracting the first equation from this result we get $d = 89$.
Since 180 is the least of the sums, $d = 89$ is the largest of the four numbers.

T-3. Given that $\angle OAB = 20°$, and $\angle OCB = 70°$, we have arc BD = 40° and arc BDE = 140°.
Then, arc DE = 100° so that $\angle EOD = 100° = \angle AOC$
Therefore, $\angle EFA = 60° = \angle CFB$, so that $\angle ABC = 50°$



T-4. Let t_o be the time to arrive precisely on time and let d be the distance traveled.
Then $d = 40\left[t_o + \left(3/60\right)\right]$ *and* $d = 60\left[t_o - \left(3/60\right)\right]$,
So, $40\left[t_o + \left(3/60\right)\right] = 60\left[t_o - \left(3/60\right)\right]$
Solving, we have $t_o = 5/20 = 1/4$ hour.
The distance traveled is $d = 40\left[\left(1/4\right) + \left(3/60\right)\right] = 10 + \left(120/60\right) = 12$ miles.
The correct average speed we want is $12 = r(t_o) = r\left(1/4\right) \Rightarrow r = 48$ miles per hour

---

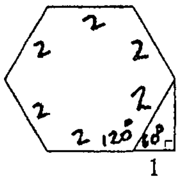
T-5. Given $\frac{a}{b} + \frac{a+10b}{b+10a} = 2$.

Divide the numerator and denominator of the second fraction by b to get $\frac{a}{b} + \frac{\frac{a}{b}+10}{1+10\left(\frac{a}{b}\right)} = 2$.

Now let $x = \frac{a}{b}$. Then the equation becomes $x + \frac{x+10}{1+10x} = 2$ and $x(1+10x) + x + 10 = 2(1+10x)$.

Simplifying, we have $10x^2 - 18x + 8 = 0 \Rightarrow (10x-8)(x-1) = 0 \Rightarrow x = 1, \frac{4}{5}$

Since $a \ne b$ then $\frac{a}{b} = \frac{4}{5}$.

1-1) 

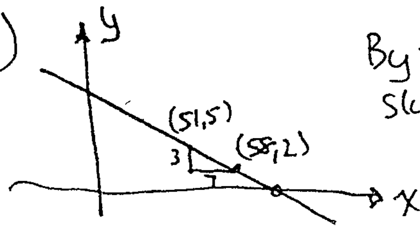
The interior angles of the regular hexagon are all 120° angles; so each side of the hexagon is twice the shorter leg of the 30°-60°-90° Δ, so each side is 2. The perimeter is $6 \times 2 = \boxed{12}$.

1-2) Since 4, 5, 6, 7, and 8 all divide $x-3$, $2^3 \cdot 7 \cdot 3 \cdot 5$ divides $x-3$ and is the smallest such divisor. Therefore 840 divides $x-3$, so $x = 840k+3$. The least such x is $\boxed{843}$.

1-3) The right side is the factorization of $g^3 - b^3$, where $g = 3x$ and $b = 2\sqrt{2}$. Thus $a = (2\sqrt{2})^2 = 8$.
More easily $-16\sqrt{2} = (-2\sqrt{2})(a)$, so $a = \boxed{8}$ by observation.
Note: The left side $27x^3 - 16\sqrt{2} = (3x)^3 - (2^{\frac{3}{2}})^3$.

1-4) The prime factors of 10 are $2 \cdot 5$. The product 25! has many more factors of 2 than of 5, so each factor of 5 can produce one factor of 10 and one consequent 0. There is a 5 every 5th number. But there's an extra 5 in 25, so the number of factors of $5 = 5+1 = \boxed{6}$ = number of terminal 0s.

1-5) $(16^{\sqrt{2}})^x$ cannot represent 1 since $x > 0$. It can equal 2 if $(16^{\sqrt{2}})^x = (2^{4\sqrt{2}})^x = 2^{4x\sqrt{2}} = 2$. That happens if $x = \boxed{\frac{1}{4\sqrt{2}}}$.

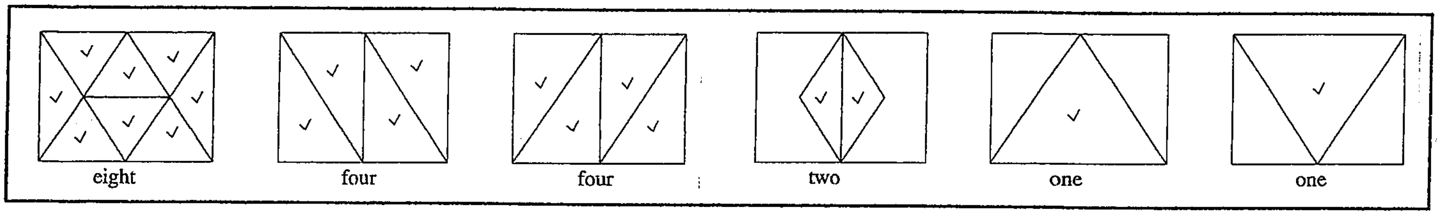
1-6) 

By trial, one solution is (58, 2). Using slope $= -\frac{3}{7}$, another solution is (51, 5). The general solution is $(58-7t, 2+3t)$. The solution with the least positive $y-x$ is $\boxed{(16, 20)}$.

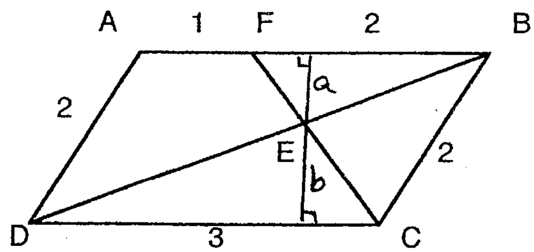
SOLUTIONS TO TEAM QUESTIONS

Contest #2, 2009 - 2010 **November 16, 2009**

T-1. There are a total of 36 triangles — the 16 smallest individual triangles plus 20 more which are shown in the diagrams below.



T-2. Let a and b represent the heights of ΔBEF and ΔCED, respectively, and let c represent the height of parallelogram ABCD. Then the height of parallelogram ABCD is equal to $c = a + b$. Since ΔCED and ΔFEB are similar, we have $\frac{a}{b} = \frac{2}{3}$, so $a = \frac{2}{3}b$. Then, the area of parallelogram ABCD is equal to $3c = 3(a+b) = 5b$, and the area of ΔCED is $\frac{3b}{2}$.



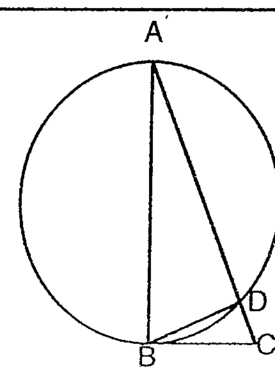
Therefore, the ratio $\dfrac{Area\ \ of\ \ \Delta CED}{Area\ \ of\ \ parallelogram\ \ ABCD} = \dfrac{3b/2}{5b} = \dfrac{3}{10}$

T-3. There are four possible combinations: (odd, odd), (even, odd), (odd, even), (even, even). The probability of "odd" and "even" are 1/2. Only the first combination produces an odd product. Therefore, the required probability is $\frac{1}{4}$

T-4. The total number of orderings is $\frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$. So there are 59 other strings.

METHOD 2: Consider that the two a's can be placed in 10 different ways. The remaining 3 letters can then be placed in $3 \cdot 2 \cdot 1 = 6$ ways. So the total number of ways is $10 \cdot 6 = 60$

T-5. Draw $\overline{BD}$. Since ΔADB is inscribed in a semicircle, then $\angle ADB = 90°$. Since it is given that $\angle BAC = 30°$, then $\angle ACB = 60°$. Therefore, we see that both ΔABC and ΔBCD are 30°-60°-90° right triangles with sides in the ratio 1:2:$\sqrt{3}$. Since it is given that $CD = \sqrt{3}$, then $BC = 2\sqrt{3}$. Then since BC is the short leg of ΔABC we find that $AB = 2\sqrt{3}\sqrt{3} = 6$



MCWP Sr. HS Mathematics League

Problem Author: Steve Conrad www.mathleague.com

Problem Editor: Dan Flegler www.mathleague.com

Brief Contest Solutions **INDIVIDUAL QUESTIONS** **Contest #3 — December 14, 2009** **2009-2010**

1) This is the Fibonacci sequence with different generators. Start with the first two terms (the generators) and add the last two to get the next. Continue forever to get

$-1, 1, 0, 1, 1, 2, 3, 5, 8, \boxed{13}, 21, 34, 55, 89, \ldots$

2) $d_1^2+d_2^2 = 4^2+7^2+4^2+7^2 = 130$. Make a list of pairs of squares of integers that have a sum of 130

$9+121=130$ so lengths 3 and 11 ← not possible since $4+7=11$

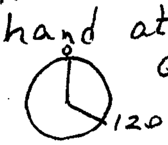
$49+81=130$ 7 and $\boxed{9}$

3) A few examples should convince you that the product of the GCD and LCM of any two positive integers is equal to the product of these two integers. See a book on number theory for a proof. the product is $\boxed{xy}$.

4 The product $6 \cdot 7 \cdot 8 \cdot 9$ must divide $x-5$. The least such product is $7 \cdot 8 \cdot 9 = 504$. Thus, $x = 504k+5$, $k \in Z^+$ and the least such integer is $\boxed{509}$.

5) $x=\sqrt{2y}$, so $x^2=2y$. Since $y=\sqrt{x}$, $2y=2\sqrt{x}$. Thus, $x^2=2\sqrt{x}$, so $x^4=4x$, so $x(x^3-4)=0$. Since $x>0$, $x=\sqrt[3]{4}$ and $y=\sqrt{x}=\sqrt{\sqrt[3]{4}}=\sqrt[3]{\sqrt{4}}=\sqrt[3]{2}$ and $(x,y)=\boxed{(\sqrt[3]{4}, \sqrt[3]{2})}$

6) At 4:00 we have the minute hand at $0°$ and the hour hand at $120°$. In 1 minute, the minute hand rotates $6°$ and the hour hand rotates $(1/2)°$. So in x minutes we want $(120+\frac{1}{2}x)-(6x)=90$. Then $5\frac{1}{2}x=30$, and $x=5\frac{5}{11}$ minutes.



SOLUTIONS TO TEAM QUESTIONS

Contest #3, 2009 - 2010 **December 14, 2009**

T-1. Let $y = 3^x$. Then the equation becomes $y - y^{-1} = \frac{80}{9} \Rightarrow 9y^2 - 9 = 80y \Rightarrow 9y^2 - 80y - 9 = 0$

Then $(9y+1)(y-9) = 0$ so, $y = 9, -\frac{1}{9}$. Now $3^x = -\frac{1}{9}$ is impossible, so $3^x = 9$ and $x = 2$

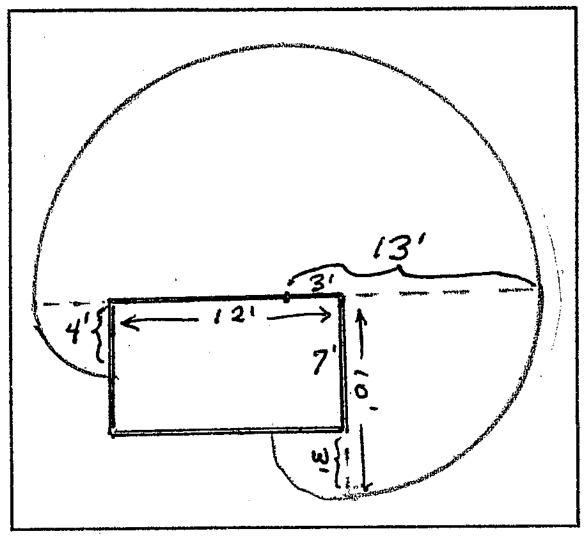
T-2. Each number can be written as $3n+1,\ n = 0,2,\ldots,38$, so $1+7+13+\ \ldots + 103+109+115$
Which has a sum of $(1+115)+(7+109)+(13+103)+\ldots = (116)(10) = 1160$

T-3. The last digits follow the same pattern as the last digits of the sequence,
$7^1 = 7,\ 7^2 = 49,\ 7^3 = 343,\ 7^4 = 2401,\ 7^5$ *ends in* $7,\ 7^6$ *ends in* $9\ \ldots$
The pattern of the last digits is: 7, 9, 3, 1, . . .
Since $\frac{778}{4}$ has a remainder of 2, the last digit of the number 777^{778} is 9.

T-4. From the diagram we see that the total roaming area is given by the areas of one semicircle of radius 13, and 3 quarter circles having radii of 4, 3, and 10. Then the dog's total roaming area is

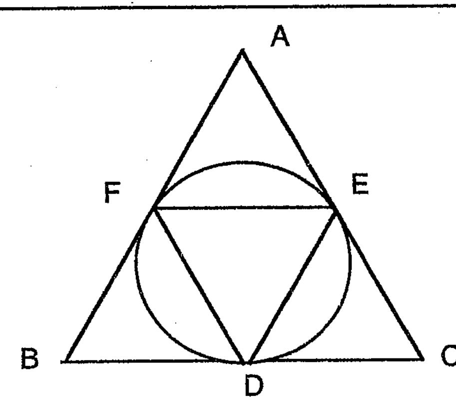
$$\pi\left(\frac{13^2}{2} + \frac{4^2}{4} + \frac{3^2}{4} + \frac{10^2}{4}\right) = \frac{\pi}{4}(169(2) + 16 + 9 + 100)$$

$$= \frac{\pi}{4}(338 + 125) = \frac{463\pi}{4}$$



T-5. The solution is obvious from the diagram – observe the 4 smaller equilateral triangles in the diagram. The area of the circumscribed triangle is 4 times the area of the inscribed triangle.

More analytically, we can use the 60° angles to determine parallel lines and show that rhombus BDEF and rhombus CDFE are congruent, and therefore, BD = DC = FE, which leads to the fact that the four smaller triangles are congruent.

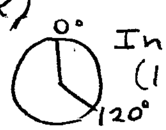


---

1) Make a chart

p	p^2-p-1
2	1
3	5
5	19
7	41

p	p^2-p-1
11	109
$\boxed{13}$	155 ← composite

2) 

In 1 min. the minute hand rotates $6°$ and the hour hand $\frac{1}{2}°$.

$(120+\frac{1}{2}x)-(6x)=90$ AND $(6x)-(120+\frac{1}{2}x)=90$

$5\frac{1}{2}x=30$ $\qquad$ $5\frac{1}{2}x=210$

$x=\frac{60}{11}=5\frac{5}{11}$ min $\qquad$ $x=\frac{420}{11}=38\frac{2}{11}$ min

So, difference is $\frac{420}{11}-\frac{60}{11}=\frac{360}{11}=32\frac{8}{11}$ min

3) $|3x|+|x+1|=2$

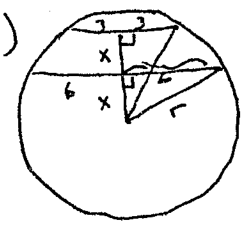
x-axis split at -1 and 0:

$x<-1$: $-3x-x-1=2$, $x=-3/4$ reject

$-1\le x<0$: $-3x+x+1=2$, $-2x=1$, $x=-\frac{1}{2}$

$x\ge 0$: $3x+x+1=2$, $4x=1$, $x=\frac{1}{4}$

$\boxed{-\frac{1}{2},\frac{1}{4}}$

4) 

$r^2=x^2+6^2=(2x)^2+3^2$

$x^2+36=4x^2+9$

$3x^2=27$

$x^2=9$, so $r^2=x^2+6^2=45$; and $\pi r^2=\boxed{45\pi}$.

5) $x+y=5 \Rightarrow x^2+2xy+y^2=25$. Now,

$xy=3 \Rightarrow x^2+6+y^2=25$. Thus

$x^2+y^2=\boxed{19}$.

6) $4^{x+2}-4^x=30$

$4^x(4^2-1)=30$

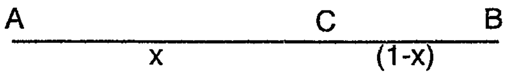
$4^x=\frac{30}{15}=2$

$x=\boxed{1/2}$.

SOLUTIONS TO TEAM QUESTIONS

Contest #4, 2009 - 2010 **January 11, 2010**

T-1.

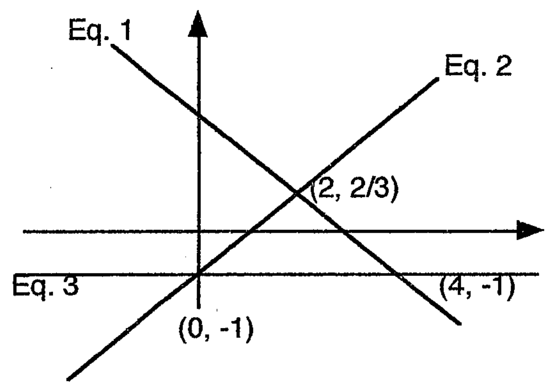


$$\frac{x}{1-x}=\frac{1}{x} \Rightarrow x^2=1-x \Rightarrow x^2+x-1=0 \Rightarrow x=\frac{-1\pm\sqrt{5}}{2}$$

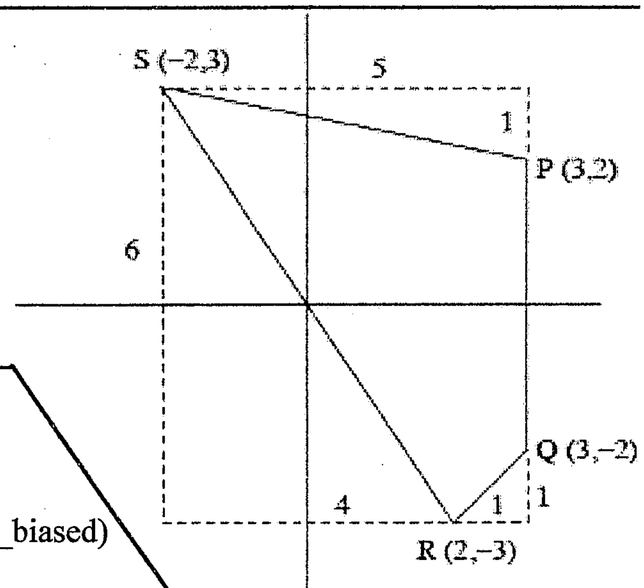
Then, the common ratio, $\frac{1}{x}=\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{2}$

T-2. Let the consecutive integers be represented by $x,\ x+1,\ x+2,\ x+3,\ and\ \ x+4$.
Then, $(x+2)+(x+3)=47 \Rightarrow x=21$
The sum of the first 5 terms is: $(x)+(x+1)+(x+2)+(x+3)+(x+4)=5x+10$
Substituting for x, we have $(5)(21)+10=105+10=115$

T-3. For discussion, label the equations:
Equation 1: $y=-\frac{5}{6}x+\frac{7}{3}$; Equation 2: $y=\frac{5}{6}x-1$ and
Equation 3: $y=-1$
Equation 3 and Equation 1 intersect at the point (4, -1).
Equation 3 and Equation 2 intersect at the point (0, -1).
Equation 1 and Equation 2 intersect at the point (2, 2/3).
So the base of the triangle has length 4, the height is $\frac{5}{3}$,
and the area of the triangle is $\frac{1}{2}\cdot 4\cdot\frac{5}{3}=\frac{10}{3}$



T-4. The Area of the rectangle is $6\times 5=30$.
Subtracting the areas of the three triangles,
we have $30-\frac{1}{2}(6\cdot 4)-\frac{1}{2}(1\cdot 5)-\frac{1}{2}(1\cdot 1)$
$30-12-\frac{5}{2}-\frac{1}{2}=15.$



T-5. Using conditional probabilities:
P(heads)=P(fair)•P(heads_fair) + P(biased)•P(heads_biased)
$=\left(\frac{3}{3+7}\right)(0.5)+\left(\frac{7}{3+7}\right)(0.6)=0.15+0.42=0.57$

MCWP Sr. HS Mathematics League

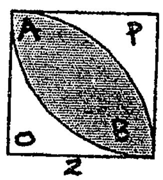
Problem Author:
Steve Conrad
www.mathleague.com

Problem Editor:
Dan Flegler
www.mathleague.com

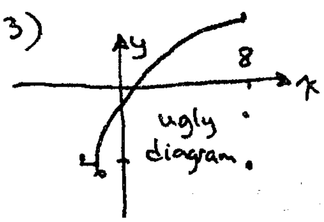
INDIVIDUAL QUESTIONS

Brief Contest Solutions **Contest #5 — February 8, 2010** **2009-2010**

1) For all b, $b>0, b\neq 1$, $\log_b b = \boxed{1}$.

2)


The side of the square has length 2. The area of the quarter-circle AOB is $\frac{1}{4}\pi r^2 = \pi$. That's also the area of the quarter-circle centered at P. Their sum is 2π. That includes all of the square plus the overlap. The area of the square is 4, so the overlap's area is $\boxed{2\pi - 4}$.

3)


The radius of the circle is 9.
The distance from $(8,-6)$ to the origin is 10.
There are no points in quadrant $\boxed{\text{II}}$

4)
$2000^x = A$ $\quad$ $A+B=9$
$2000^y = B$ $\quad$ $AB = 8$ $\quad$ $\therefore (A,B) = (1,8)$ or $(8,1)$

$2000^{2x} = A^2$ $\quad$ $A^2+B^2 = (A+B)^2 - 2xy = 81-16 = \boxed{65}$.
$2000^{2y} = B^2$ $\quad$ (or $A^2+B^2 = 1^2+8^2 = 65$)

5)
$$\begin{array}{r l}
 & x \quad + \quad 7 \\
x^2+1 \; \big) & x^3 + 7x^2 + Ax + B \\
 & x^3 \qquad\quad + x \\
\hline
 & 7x^2 + (A-1)x + B \\
 & 7x^2 \qquad\qquad + 7
\end{array}$$

Now, the remainder is $(A-1)x + (B-7)$. Thus, $(A,B) = \boxed{(1,7)}$.

6) Method I (can you explain this method?)
$19+18+17+\ldots+3+2+1$
$= \frac{19}{2}(20) = \boxed{190}$.

Method 2 $\binom{20}{2} = 190$ $\quad$ Method III $\frac{20\times19}{2} = 190$

Explanation of Method I: Man 20 shakes everyone else's hand, all 19, then leaves. Same for man 19, who shakes 18 other hands.

SOLUTIONS TO TEAM QUESTIONS

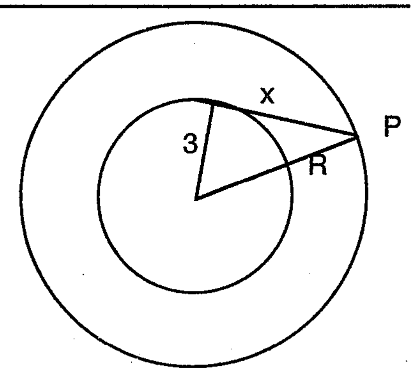
Contest #5, 2009 - 2010 **February 8, 2010**

T-1. The number of combinations is given by $\frac{4!}{2!} = 4 \times 3 = 12$.
So there are 11 other integers that can be formed.

T-2. Note that x cannot equal -1. Then consider 2 cases:
If $x > -1$, then we have $\frac{2x+5}{|x+1|} = \frac{2x+5}{x+1} \geq 1 \Rightarrow 2x+5 \geq x+1 \Rightarrow x \geq -4$
So, the inequality is satisfied for all $x > -1$.
If $x < -1$, then we have $\frac{2x+5}{|x+1|} = \frac{2x+5}{-x-1} \geq 1 \Rightarrow 2x+5 \geq -x-1 \Rightarrow 3x \geq -6 \Rightarrow x \geq -2$
So, the inequality is satisfied for all x *where* $-2 \leq x < -1$.
Combining these 2 solutions, we see that the inequality is satisfied for all x such that
$-2 \leq x < -1 \cup x > -1$

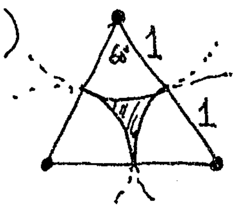
T-3. Substitute for $z = -x - y$, so that $x^3 + y^3 + z^3 = x^3 + y^3 + (-x-y)^3$
$= x^3 + y^3 - x^3 - 3x^2y - 3xy^2 - y^3 = -3x^2y - 3xy^2 = 3xy(-x-y) = 3xyz = 3(2) = 6$

T-4. Let R be the radius of the larger circle. Then
$2\pi(3)^2 = \pi R^2 \Rightarrow 18\pi = \pi R^2 \Rightarrow R^2 = 18$
Now, by the Pythagorean Theorem, $x^2 = R^2 - 3^2 = 18 - 9 = 9$
So, $x = 3$



T-5. Let S_1, S_2, S_3, S_4, *and* S_5 represent the amount spent in stores 1, 2, 3, 4, and 5.
To start, the amount of Albert's money, M, is $M = S_1 + S_2 +_3 S + S_4 + S_5$. Working backward:
In the fifth store, we have $S_5 = \left(\frac{1}{2}\right)(M - S_1 - S_2 - S_3 - S_4) + 1 = \left(\frac{1}{2}\right)S_5 + 1$, so $S_5 = 2$
In the fourth store, we have $S_4 = \left(\frac{1}{2}\right)(M - S_1 - S_2 - S_3) + 1 = \left(\frac{1}{2}\right)(S_4 + S_5) + 1$
then $\left(\frac{1}{2}\right)S_4 = S_5 + 1 = 2$, so $S_4 = 4$
In the third store, we have $S_3 = \left(\frac{1}{2}\right)(M - S_1 - S_2) + 1 = \left(\frac{1}{2}\right)(S_3 + S_4 + S_5) + 1$
then $\left(\frac{1}{2}\right)S_3 = \left(\frac{1}{2}\right)(2+4) + 1$, so $S_3 = 8$
In the second store, we have $S_2 = \left(\frac{1}{2}\right)(M - S_1) + 1 = \left(\frac{1}{2}\right)(S_2 + S_3 + S_4 + S_5) + 1$
then $\left(\frac{1}{2}\right)S_2 = \left(\frac{1}{2}\right)(2+4+8) + 1$, so $S_2 = 16$
So, in the first store, we have $S_1 = \left(\frac{1}{2}\right)(M) + 1 = \left(\frac{1}{2}\right)(S_1 + S_2 + S_3 + S_4 + S_5) + 1$
then $\left(\frac{1}{2}\right)S_1 = \left(\frac{1}{2}\right)(2+4+8+16) + 1$, so $S_1 = 32$
Then $M = S_1 + S_2 +_3 S + S_4 + S_5 = 62$

1)



$r=1$

$s=2$

$\text{Area}_{\triangle} = \dfrac{s^2\sqrt{3}}{4} = \sqrt{3}$

$3\left(\tfrac{1}{6}\text{ circle}\right) = \dfrac{\text{circle}}{2} = \dfrac{\pi}{2}$

Shaded region $= \triangle - \text{half-circle}$

$= \boxed{\sqrt{3} - \frac{\pi}{2}}$.

2)

$$\begin{aligned} 269x + 231y &= 288 \\ 231x + 269y &= 212 \end{aligned}$$

Add: $500x + 500y = 500$ or $x+y=1$

Subtract: $38x - 38y = 76$ or $x - y = 2$

$$\begin{aligned} x+y&=1 \\ x-y&=2 \\ 2x &= 3 \\ x &= \tfrac{3}{2} \end{aligned}$$

$(x,y) = \boxed{\left(\frac{3}{2}, -\frac{1}{2}\right)}$

3) Avg rate $= \dfrac{\text{Total distance}}{\text{total time}}$

one-way dist $= d$

$6 = \dfrac{2d}{\frac{d}{4} + \frac{d}{k}} = \dfrac{8dk}{d(k+4)} = \dfrac{8k}{k+4}$

$\therefore\ 6k + 24 = 8k$

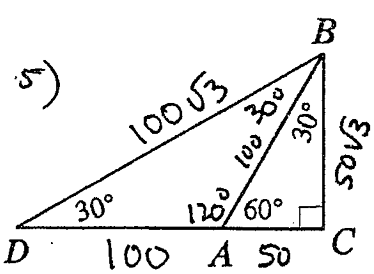
$k = \boxed{12}$

4) $(r+3)(10) = (3-r)(50)$

$r=$ rate, in steps per second

$r = \boxed{2}$

5)



$\boxed{50\sqrt{3}}$

6) $\log_b a = \dfrac{\log_c a}{\log_c b} \Rightarrow (\log_{16} 9)(\log_3 25)(\log_5 4) = \dfrac{\log 9}{\log 16} \cdot \dfrac{\log 25}{\log 3} \cdot \dfrac{\log 4}{\log 5}$

if $a>0$
$b>0,\ b\neq 1$
$c>0,\ c\neq 1$

$= \dfrac{2\cancel{\log 3}}{4\cancel{\log 2}} \cdot \dfrac{2\cancel{\log 5}}{\cancel{\log 3}} \cdot \dfrac{2\cancel{\log 2}}{\cancel{\log 5}}$

$= \boxed{2}$.

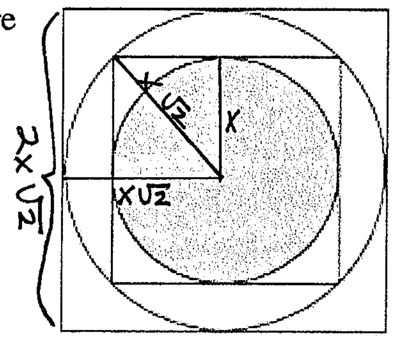
SOLUTIONS TO TEAM QUESTIONS

Contest #6, 2009 - 2010 **March 8, 2010**

T-1. The numbers in geometric progression have the form, $a,\ ar,\ ar^2$, so we have
$(a)(ar)\left(ar^2\right)=216=6^3$, so $(ar)^3=6^3$ *and* $ar=6$, *so then* $a=\frac{6}{r}$.
Also, $a+ar+ar^2=19 \Rightarrow a\left(1+r+r^2\right)=19 \Rightarrow \left(\frac{6}{r}\right)\left(1+r+r^2\right)=19$, so that
$6+6r+6r^2=19r \Rightarrow 6r^2-13r+6=0 \Rightarrow (3r-2)(2r-3)=0$, and $r=\frac{3}{2},\ \frac{2}{3}$.
For $r=\frac{3}{2}$, we have the numbers $4, 6, 9$. For $r=\frac{2}{3}$, we have the numbers $9, 6, 4$.
So the largest number is 9.

T-2. 10^{55} ends in 55 zeros. Subtracting 55, we have $10^{55}-55=999\cdots945$, where the first 53 digits are 9's. So, the sum of the digits is $53(9)+9=54(9)=486$

T-3. If x is the radius (in inches) of the shaded circle, then the larger square has length $2x\sqrt{2}$. Then $\left(2x\sqrt{2}\right)^2=8x^2=108$.
Then we have $x^2=\frac{27}{2}$.
So, the area of the shaded circle is $\pi x^2=\frac{27\pi}{2}$ square inches.



T-4. $a_4+a_8+a_{12}+a_{16}=(a_1+3d)+(a_1+7d)+(a_1+11d)+(a_1+15d)$.
$=4a_1+36d=4(a_1+9d)=224$, so $a_1+9d=56$.
Now, the sum of the first 19 terms is $a_1+(a_1+d)+(a_1+2d)+\cdots+(a_1+17d)+(a_1+18d)$
$=19a_1+d\left(\frac{18\times 19}{2}\right)=19a_1+9(19)d=19(a_1+9d)=19(56)=1064$.

T-5. Consider the possibilities. For example, 4+4+2=10 and $4\cdot 4\cdot 2=32$.
The largest possible product is given by $2\cdot 2\cdot 3\cdot 3=36$
